Pergamon

Int. J. Solids Structures Vol. 34, No. 28, pp. 3657–3672, 1997

0020–7683/97 $17.00 + .00

PII: S0020–7683(96)00229–6

ORTHOGONAL LOAD RESOLUTION AND STATICAL-KINEMATIC STIFFNESS MATRIX

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(*Received* 20 *March* 1996; *in revised form* 4 *November* 1996)

Abstract—The structural behavior of underconstrained systems is explored with an emphasis on statically indeterminate ones. The roles of the elastic stiffness and the statical-kinematic stiffness (induced by prestress or an equilibrium load) are investigated and compared. The linearized response of a prestressed underconstrained system is shown to go beyond the simple notion of the first-order stiffness induced by prestress and independent of the member stiffnesses. Some interesting features of structural behavior under a general load and stability under an equilibrium load are revealed and illustrated with an example of a quasi-variant pin-bar system. Finally, a comprehensive stiffness matrix with coefficients representing various sources of first-order stiffness is introduced, to replace, when necessary, the conventional elastic stiffness matrix. The comprehensive matrix is valid for structural systems of any type and is always invertible. © 1997 Elsevier Science Ltd.

INTRODUCTION

A structural system that is not geometrically invariant (an underconstrained system) generally possesses kinematic mobility: it allows finite displacements without any deformations of it members. However, some exceptional underconstrained systems possess only infinitesimal mobility; more precisely, their first-order infinitesimal displacements are possible at the expense of higher-order strains. Such a system (an infinitesimal mechanism) can be defined as a system allowing virtual displacements but no kinematic displacements. Some, but not all, systems with infinitesimal mobility are prestressable and found wide applications in engineering practice.

Analytical statics of underconstrained structural systems, including prestressable ones, originated in the works of Maxwell (1890), Mohr (1885) and Levi-Civita and Amaldi (1930). These classical works are the basis for most of the subsequent investigations. Typical of the later contributions to the subject is the extensive and efficient use of the modern matrix analysis (Calladine, 1982; Pellegrino and Calladine, 1986, 1991). Here, as well as in Kuznetsov (1991), the statical-kinematic and the elasticity aspects of the problem are uncoupled and treated more or less independently. On the other hand, Volokh and Vilnay (in press) employ a more comprehensive problem statement, typical of structural mechanics, including stability and vibration issues. They show, in particular, how and when the above uncoupling takes place.

The main emphasis of this paper is on the implications of statical indeterminacy in underconstrained systems, especially prestressable ones. The analytical background as well as the format of the following development are due to Levi-Civita and Amaldi (1930).

Complete information on the kinematic properties of a system with ideal (undeformable, inextensible) positional constraints is contained in a set of simultaneous constraint equations,

$$F^i(X_1, \ldots, X_n, \ldots, X_N) = 0, \quad i = 1, 2, \ldots, C. \tag{1}$$

The C constraint functions F^i relate the N generalized coordinates, X_n, to the invariant geometric parameters of the system (member lengths, angular distances, etc.) implicitly present in the equations. At least one solution to the constraint equations, $X_n = X_n^0$, is assumed to be known and is taken as the reference geometric configuration. If the solution is not an isolated point, the system possesses kinematic (finite) mobility. Otherwise, the system has a unique geometric configuration although it may still possess virtual (infinitesimal) mobility. Further investigation requires expanding the functions F^i into power series at the solution point $X_n = X_n^0$:

$$F^i_n x_n + (1/2!) F^i_{mn} x_m x_n + \cdots = 0, \quad m, n = 1, 2, \ldots, N. \tag{2}$$

Here x_n are infinitesimal increments of the respective coordinates (that is, virtual displacements of the system) and a repeated index denotes summation over the indicated range.

The linear terms of the expansion appear in the linearized constraint equations

$$F^i_n x_n = 0, \tag{3}$$

where the first derivatives,

$$F^i_n = \partial F^i / \partial X_{n|0}, \tag{4}$$

are the elements of the constraint function Jacobian matrix at X_n^0.

The Jacobian matrix rank r being equal to N is an analytical criterion of a geometrically invariant system; at $r < N$ the system is underconstrained. In the latter case, eqns (3) can be solved in terms of properly selected $V = (N-r)$ virtual displacements x_s chosen as independent. By giving each of them, one at a time, a unit magnitude while keeping the rest of x_s equal to zero, all virtual displacements of the system can be expressed with the aid of an $N \times V$ matrix a_{ns}:

$$x_n = a_{ns} x_s, \quad a_{ns|n=s,t} = I_{st} \tag{5}$$

where I_{st} is the identity matrix. The subscripts s and t cover one and the same range of size V and designate the independent displacements. Any particular set of virtual displacements of the system is obtained by arbitrarily specifying the magnitudes of the independent displacements x_s serving as the scale factors in eqn (5).

As shown in analytical statics, complementing the principle of virtual work with the linearized constraint eqn (3) for a structural system leads to the equilibrium equations

$$F^i_n \Lambda_i = P^*_n. \tag{6}$$

It is readily seen from the respective index patterns in (3) and (6) that the equilibrium matrix is the transpose of the constraint Jacobian matrix. The generalized constraint reactions, Λ_i, depend on the particular form of the constraint functions (1), and do not necessarily coincide with the actual member forces in the system. For a pin-bar assembly, Λ_i will be identical to the bar forces if the inextensibility conditions for the bars are represented by constraint equations of the form

$$\sqrt{\Delta X^2 + \Delta Y^2 + \Delta Z^2} - L = 0 \tag{7}$$

where $\Delta W = W_{\text{far}} - W_{\text{near}} = L_w$ $(W = X, Y, Z)$ is the coordinate distance between the far and the near ends of the bar, i.e., the length of the bar projection on the respective coordinate axis. After denoting the projection angles as θ_w, the Jacobian matrix elements acquire the form

$$L_w / L = \pm \cos \theta_w \tag{8}$$

and are easily recognized as the elements of the equilibrium matrix for the system. Thus, for a system with ideal constraints in a known geometric configuration, the construction of the constraint Jacobian matrix, hence, of the linearized constraint equations, does not require writing down, or even knowing explicitly, the constraint equations.

ORTHOGONAL LOAD RESOLUTION

In any given configuration, an underconstrained system can balance only those loads P_n^* that lie in the column space of the equilibrium matrix. These statically possible loads (equilibrium loads) must be linear combinations of the matrix columns; hence, the number of independent equilibrium loads equals the matrix rank, r. For a system with $r = N$, and only for such a system, any load is an equilibrium load (the statical criterion of geometric invariance).

When $r < N$, the equilibrium eqns (6) with a general load P_n in the right hand side become incompatible and can be solved only approximately. An approximate solution does not satisfy the equations, producing instead an error

$$E_n = P_n - F_n^i \Lambda_i. \tag{9}$$

The best solution in the sense of the minimum mean square error is one with the error vector E_n orthogonal to the column space of the equilibrium matrix, i.e., one satisfying

$$F_n^j E_n = F_n^j (F_n^i \Lambda_i - P_n) = 0. \tag{10}$$

Since the matrix rank is $r \leqslant C$ (recall that C is the number of constraints), only r reactions can be found from (10). This is done by choosing a statically determinate subsystem of the original system and retaining in the matrix F_n^i in (10) the corresponding set of columns of rank r:

$$\Lambda_p = (F_o^p F_o^q)^{-1} F_n^q P_n, \quad m, n, o = 1,2, \ldots, N. \tag{11}$$

The subscripts p and q span one and the same range and designate the r constraint reactions of the statically determinate subsystem. With the constraint reactions (11), it equilibrates a load

$$P_n^* = F_n^p (F_o^p F_o^q)^{-1} F_m^q P_m \equiv R_{nm} P_m \tag{12}$$

which is in the column space of the equilibrium matrix. The matrix operator R_{mn} (projection matrix, Strang, 1988), projects a general load P_n onto the column space of the equilibrium matrix.

At $r < C$ solution (11) provides just one of the statically possible force distributions in the system; specifically, the one in the statically determinate subsystem comprised of the retained r bars. Nevertheless, the equilibrium part P_n^* of the load P_n is uniquely and correctly determined by (12) regardless of the composition of the statically determinate subsystem; the presence of dependent (redundant) constraints does not affect the equilibrium matrix column space.

Note that P_n^* is the complete equilibrium part of the load P_n; the remaining part [see (9)],

$$p_n = P_n - P_n^* = E_n, \tag{13}$$

is a pure perturbation load (Kuznetsov, 1991). Perturbation loads are not only orthogonal (as is the vector E_n) to the column space of the equilibrium matrix, but span its orthogonal complement space. The latter happens to be the null space of the constraint Jacobian matrix, i.e., the space of the inextensible virtual displacements a_{ns}. Taking advantage of this fact (a manifestation of the statical-kinematic duality), a general perturbation load can be represented by the same matrix as the virtual displacements:

$$p_{ns} = a_{ns}. \tag{14}$$

For a given general load P_n, the $V = (N-r)$ independent components p_s of the perturbation load are evaluated using the equation of virtual work together with relations (5) and (14). Setting to zero the work of the equilibrium load P_n^* over the virtual displacements x_n gives

$$P_n^* x_n = (P_n - p_n)x_n = (P_n - a_{nt}p_t)a_{ns}x_s = 0. \tag{15}$$

Since this equation must be satisfied identically with respect to all of the independent virtual displacements x_s, it yields V equations in the unknown independent components p_t of the perturbation load p_n contained in P_n:

$$a_{sn}a_{nt}p_t = a_{sn}P_n. \tag{16}$$

Equation (16) can be used to identify an equilibrium load: if, and only if, this matrix product in the right hand side vanishes, then $p_t = 0$ and the entire load is an equilibrium load: $P_n = P_n^*$. More importantly, eqn (16) leads to an explicit evaluation, first, of the independent components p_t and then, with the aid of (5), of the entire perturbation load p_n contained in P_n:

$$p_t = (a_{tn}a_{ns})^{-1}a_{sm}P_m, \tag{17}$$

$$p_n = a_{ns}(a_{so}a_{ot})^{-1}a_{tm}P_m \equiv r_{nm}P_m. \tag{18}$$

The introduced matrix operator r_{mn} projects a general load P_n onto the left null space of the equilibrium matrix (the null space of the constraint Jacobian matrix, hence, the space of the virtual displacements of the system).

The condition of orthogonality between perturbation loads p_n and the column space of the equilibrium matrix F_n^i provides a means for identifying a perturbation load: if, and only if,

$$F_n^i P_n = 0, \tag{19}$$

the load P_n does not contain an equilibrium part and is a perturbation load: $P_n = p_n$.

The most important properties of the two loads can be summed up as follows. An equilibrium load may be balanced by the system in the given configuration without any displacements (as long as the constraints are ideal). Yet, it is just a statically possible load, and the attained equilibrium is not necessarily stable. A perturbation load cannot be equilibrated by the system in its original configuration; it produces either infinite constraint reactions (in systems with infinitesimal mobility) or sets the system in motion. Remarkably, a perturbation load does not induce any internal forces in the system and is counteracted solely by the D'Alembert (inertia) forces of the incipient motion. This definitive property of perturbation loads is relevant in the context of the optimal motion control (minimum energy requirement for generating the control forces).

For a system in a given geometric configuration subjected to a general load P_n, eqns (12) and (18) determine, respectively, the equilibrium and perturbation parts of the load such that

$$P_n^* + p_n = P_n, \quad P_n^* p_n = 0. \tag{20}$$

The orthogonal load resolution (20) always exists, is unique, and configuration-specific. Its implementation is the key to the analysis of underconstrained structural systems.

The ranks of the projection matrices (12) and (18) are, respectively, r and $V = (N-r)$; their sum, according to (20_1), is the identity matrix:

$$R_{mn}+r_{mn} = I_{mn}. \tag{21}$$

Thus, evaluating either one of the projection matrices suffices for carrying out the load resolution. The second alternative (the matrix r_{mn}) appears advantageous for two reasons: one is a lower computational expense of the matrix operations, determined by a typical relation between the numbers C, N, and r; the other is the independent need for using the intermediate matrix a_{ns} when evaluating the system displacements.

STATICAL INDETERMINACY

Whereas the degree of virtual mobility, $V = N-r$, is always positive, the degree of statical indeterminacy of an underconstrained system, $S = C-r$, can be either positive or zero. At $r = C$ the system is statically determinate, and the bar force distribution (11) is unique. In fact, the statical analysis may be terminated at this point: indeed, if finding the approximate bar forces is the only objective, there is no need for the orthogonal load resolution or even for the matrix operation (12) explicitly evaluating P_n^*. However, there are at least two compelling reasons for carrying out the load resolution. First, it is necessary for assessing the accuracy of the obtained bar forces, determined solely by the equilibrium part of the given load. Second, calculating the system displacements requires knowing the perturbation part of the load (the cause of the displacements) while the matrix quantifying the system resistance to such loads (the statical-kinematic stiffness matrix) depends on the bar forces induced by the equilibrium part of the load.

At $r < C$ the structural system is statically indeterminate and, beyond the load resolution, no progress can be made in the analysis under the assumption of ideal (undeformable) constraints. Evaluating the stress state under the equilibrium load, necessary for constructing the statical-kinematic stiffness matrix, requires introducing the elastic properties of the structural members.

Consistent with the format of the foregoing development, analysis of statical indeterminacy is based on the incremented constraint eqns (1):

$$F^i_n x_n = f_i. \tag{22}$$

Here the constraint variations f_i in the right hand side represent small changes in the geometric parameters of the system (say, the bar lengths) due to elastic, plastic, rheological, thermal, and other possible deformations of the structural members. Since the Jacobian matrix rank is $r < C$, a general set of constraint variations f_i can be resolved into two mutually orthogonal subsets—compatible variations and anticompatible ones, residing, respectively in the column space of the Jacobian matrix and in its orthogonal complement space. The latter space, by virtue of the statical-kinematic duality, is the null space of the equilibrium matrix, i.e. the space of the $S = C-r$ linearly independent states of self-stress. By choosing one of the reactions in each state as a scale factor and giving it a unit magnitude, all of the self-equilibrated constraint reactions can be expressed as a $C \times S$ matrix b_{ik} with $b_{ik|i=k,l} = I_{kl}$. The subscripts k and l cover one and the same range of the size S and designate those of constraint reactions Λ_i that are chosen as independent. Any particular state of self-stress of the system is obtained by arbitrarily specifying the magnitudes of Λ_k in the equation

$$\Lambda_i = b_{ik}\Lambda_k. \tag{23}$$

Taking advantage of the statical-kinematic duality, a general anticompatible constraint variation can be represented by the same matrix as the self-equilibrated constraint reactions:

$$f^{\perp}_{ik} = b_{ik}. \tag{24}$$

The statical indeterminacy is resolved by taking advantage of orthogonality between the actual constraint variations, which must be compatible, and the anticompatible ones from (24):

$$(f_i^r + f_i) f_{ik}^{\perp} = 0. \tag{25}$$

The actual variations (parenthesized) are the elastic deformations of the structural members. The first term represents the deformations induced by the applied equilibrium load in the chosen statically determinate subsystem with r constraints; the deformations of the redundant constraints in this array (those with subscript k) are set to zero. The second term describes the constraint deformations due to the yet unknown redundant forces of each of the S self-stress states. Since matrix (24) spans the subspace of anticompatible constraint variations, any matrix orthogonal to it, in particular, the parenthesized matrix in (25), represents a compatible constraint variation.

For a linearly elastic pin-bar system, the deformations (bar elongations) obey Hooke's law:

$$f_i = D_{ij}\Lambda_j, \quad D_{ij} = (L/EA)_{ii}. \tag{26}$$

The elements of the diagonal flexibility matrix D_{ij} are functions of the i-th bar length, modulus of elasticity and cross sectional area.

The magnitudes Λ_k of the independent redundant constraint reactions are evaluated from a simultaneous system of equations obtained by combining eqns (23) through (26):

$$(D_{ij}\Lambda_j^r + D_{ij} b_{jk}\Lambda_k) b_{il} = 0 \tag{27}$$

or, after factoring out the member flexibility matrix,

$$D_{ij}(\Lambda_j^r + b_{jk}\Lambda_k) b_{il} = 0. \tag{28}$$

Here Λ_i^r are the constraint reactions in the statically determinate subsystem. The only free subscript, l, is identical in its range, S, to the subscript k; thus, S is the number of both the simultaneous equations and unknowns in (28).

Upon evaluating Λ_k, the member forces in the system are obtained in a usual way:

$$\Lambda_i = \Lambda_i^r + b_{ik}\Lambda_k. \tag{29}$$

Remarkably, eqn (28) is devoid of any notion of the statical-kinematic type of the system in consideration. Thus, it is a general equation for analysis of statical indeterminacy of any structural system, geometrically invariant or underconstrained. The condition of orthogonality underlying the development of this equation is a general form of the compatibility condition in statically indeterminate analysis. In particular, the condition of orthogonality, as implemented in eqn (27), is easily adapted for a temperature problem by replacing the elastic deformations in the statically determinate subsystem with the thermal deformations of all system members:

$$(f_i^0 + D_{ij} b_{jk}\Lambda_k) b_{il} = 0, \quad f_i^0 = (\alpha TL)_i. \tag{30}$$

Here α is the thermal expansion factor and T is the temperature change of the i-th member.

Note that the introduction of constraint variations (whether elastic or thermal) into the right hand side of the linearized constraint eqns (22) does not refine the geometric configuration of the system and the analysis remains geometrically linear. The situation is the same as in any conventional problem where elastic displacements are evaluated upon determining the member forces. A change in the geometric configuration as well as the bar elongations can be accounted for only by modifying the nodal coordinates and bar lengths in the original constraint eqns (1).

THE STATICAL-KINEMATIC STIFFNESS MATRIX

An underconstrained structural system in a state of equilibrium under an external load and/or initial forces (prestress) obeys equilibrium eqn (6). If this state is subjected to a small perturbation, the equation must be varied by incrementing all terms in both sides:

$$\delta(F^i_n\Lambda_i) \equiv F^i_{nm}\Lambda_i x_m + F^i_n\lambda_i = \delta P_n. \tag{31}$$

Here λ_i are small increments of the respective constraint reactions Λ_i, and the right hand side represents an additional external load which is assumed arbitrary. The variation symbol δ in the right hand side implies independent incrementation of all components of the previously applied load. Having in mind that the original load and forces of prestress are already reflected in the values of Λ_i, the symbol δ is omitted in the further analysis.

The symmetric matrix

$$K_{mn} = F^i_{mn}\Lambda_i \tag{32}$$

is the complete statical-kinematic stiffness matrix of a structural system. By premultiplying displacements x_m in (31) it evaluates the changes in the nodal resultants of the member forces due to a change in the geometric configuration of the system (much like the stiffness matrix accounting for the presence of prestress forces). The matrix is assembled of the constraint function Hessian matrices (weighed by Λ_i) appearing in the power series expansion (2) as the second-order term coefficients. The Hessian matrix for a constraint function of the form (7) is utterly simple: its only nonzero elements are those with the subscripts m and n present in the function, with the diagonal elements ($m = n$) equal to L_i^{-1} and the off-diagonal ones equal to $-L_i^{-1}$.

It is possible to eliminate λ_i from (31), leaving displacements as the only unknowns. To this end, both sides are multiplied by x_n and, in accordance with (3), the second product vanishes from the resulting equation of virtual work. Next, displacements x_m are expressed in terms of the independent displacements with the aid of (5). Since the equation of virtual work must be satisfied identically with respect to each of the independent virtual displacements, it yields V simultaneous equilibrium equations in the unknown independent displacements x_t:

$$a_{sm}K_{mn}a_{nt}x_t = a_{sn}P_n. \tag{33}$$

Introducing a $V \times V$ matrix,

$$K_{st} = a_{sm}K_{mn}a_{nt}, \tag{34}$$

allows the system of equilibrium equations to be rewritten as

$$K_{st}x_t = a_{sn}P_n. \tag{35}$$

The symmetric matrix K_{st} derived from K_{mn} is the reduced statical-kinematic stiffness matrix. This matrix premultiplies the independent displacements to evaluate the generalized nodal force resultants that equilibrate the generalized load in the right hand side of (35) [cf. eqn (16)]. Note that the latter load, being a generalized perturbation load, is not affected by the equilibrium part of the given load; formally, this follows from the orthogonality condition between virtual displacements and equilibrium loads,

$$a_{sn}P^*_n = 0. \tag{36}$$

The above equation represents the definition of an equilibrium load in terms of the principle of virtual work.

Although the equilibrium part of the applied load does not give rise to displacements, if affects the system displacements indirectly, by contributing to the constraint reactions Λ_i and, thereby [see (32)], altering the system resistance to the perturbation part of the load. It should be noted that constraint reactions produced by an equilibrium load do not necessarily induce stiffness in the system and may even destabilize its initial stress state, as will be demonstrated in the numerical study in the next section.

In this light, the proper course of action in evaluating the system displacements is: separate the equilibrium part of the applied load; find the resulting constraint reactions (solving, perhaps, a statically indeterminate problem); add them to the previously existing ones; and incorporate the result into (32).

The main features of the reduced statical-kinematic stiffness matrix K_{st} can be summed up as follows.

(1) The matrix quantifies the resistance of the system with idealized (perfectly rigid) members to perturbation loads, to which it relates the system displacements. Thus, the very concept of statical-kinematic stiffness is meaningful only for underconstrained structural systems (geometrically invariant systems do not have perturbation loads).

(2) The matrix depends solely on the member forces and geometric configuration of the system. The origin of the member forces (prestress, thermal stresses, equilibrium load) is irrelevant.

(3) Positive definiteness of the reduced matrix indicates that the system in the given state of equilibrium resists small perturbations. This is a necessary and sufficient criterion of geometric stability of the equilibrium state. It also means that the complete matrix (32) is positive definite, subject to linearized constraint conditions (3), i.e., in the space of virtual displacements.

(4) The fact that solution of eqn (35) must be preceded by the orthogonal load resolution, evaluation of the member forces, and their incorporation into K_{st} underlies the unusual response of an underconstrained system to a general (non-equilibrium) load. For an initially stress-free system, the displacements depend on the load pattern but not the magnitude (the latter affects equally both sides of the equation). For a system with pre-existing internal forces (of any origin), the two sides of eqn (35) are affected by the load magnitude differently, thus leading to a non-proportional load-displacement relation resembling that of a beam-column or frame analysis.

(5) Elastic properties of the structural members, seemingly absent in the statical-kinematic stiffness matrix, may affect it implicitly, by way of the member forces. Specifically, member forces in a statically indeterminate system under an equilibrium load depend on the member stiffness ratios. On the other hand, member forces in a statically determinate system, as well as forces of prestress in a statically indeterminate system, are independent of the elastic properties.

Aside from the described features, the statical-kinematic stiffness matrix is similar to the conventional elastic stiffness matrix employed in a linear or linearized incremental analysis of geometrically invariant structural systems.

To fully account for the elastic properties of the system, eqns (31) must be used together with the linearized inhomogeneous constraint eqns (22), where the constraint variations f_i represent the elastic and thermal deformations of the structural members. The elastic deformations are expressed in terms of λ_i as in (26), whereas the thermal deformations, assumed known, are retained in the right hand side of (22). The resulting combined system of equations can be presented in a block-matrix form:

$$\begin{bmatrix} K_{nm} & F_n^j \\ F_m^i & -D_{ij} \end{bmatrix} \begin{bmatrix} x_m \\ \lambda_j \end{bmatrix} = \begin{bmatrix} P_n \\ f_i^0 \end{bmatrix}. \tag{37}$$

In order to eliminate the λ_i from this system of equations, both sides of the lower block are multiplied by the diagonal matrix of member stiffnesses, C_{ij} (the inverse of the member flexibility matrix D_{ij}), to give

$$\lambda_i = C_{ij}(F^j_m x_m - f^0_j), \quad C_{ij} = (EA/L)_{ii}. \tag{38}$$

This result is introduced into the upper part of (37):

$$K_{nm}x_m + F^i_n C_{ij}(F^j_m x_m - f^0_j) = P_n. \tag{39}$$

The combination of matrices that premultiply displacements x_m in this equation is the comprehensive stiffness matrix of a structural system. It is the sum of the complete statical-kinematic stiffness matrix (32) and the elastic stiffness matrix of the system:

$$K^c_{mn} = K^{sk}_{mn} + K^e_{mn} = F^i_{mn}\Lambda_i + F^i_m C_{ij} F^j_n. \tag{40}$$

For underconstrained systems, the matrix superposition remedies singularity of the elastic stiffness matrix by providing a solution within its null space (the space of inextensible displacements).

The presented analytical development underlies several distinct levels in the statical analysis of underconstrained structural systems. These are described below in the ascending order of complexity which is adhered to in the example in the next section.

Level 1. The simplest analysis is confined to determining the member forces in a system subjected to a general load. For a statically determinate system ($r = C \leqslant N$), the member forces are immediately obtainable with the aid of (11). If the system is statically indeterminate, the matrix b_{ik} describing the $C-r$ self-stress states must be constructed first. Constraint reactions in the chosen statically determinate subsystem can be found from (11). After forming and solving the system of eqns (28), the member forces are evaluated according to (29).

The orthogonal load resolution is present in this analysis, at least implicitly [cf. (11) and (12)], but the statical-kinematic stiffness matrix does not figure in it at all, making it impossible to assess stability of the obtained equilibrium state or even the accuracy of the results. Thus, the analysis is appropriate only when both of these issues are of no concern, e.g., a small perturbation of a highly prestressed tensile system where stability is obvious and only small displacements are expected.

Level 2. Statical-kinematic, i.e., geometric, stability of the found equilibrium state is verified by evaluating the reduced statical-kinematic stiffness matrix (34) and checking it for positive definiteness either in two dimensions (plane structure) or in three dimensions.

Level 3. Displacements of an underconstrained system with idealized (inextensible) members under a general load are evaluated by solving eqn (35) with a given load in the right hand side. The displacement magnitudes are useful in assessing the accuracy of the performed linear analysis.

Level 4. By taking into account the elastic properties of the system, this analysis refines the displacements (but not the member forces) obtained at the level 3. The comprehensive stiffness matrix (40) employed in this analysis is invertible for an elastic system of any type under any load. The obtained results can be improved only by carrying out a geometrically nonlinear analysis.

EXAMPLE, OBSERVATIONS AND DISCUSSION

Consider a plane underconstrained pin-bar system consisting of two affine symmetric pressure polygons connected with three vertical bars (Fig. 1). Its geometric configuration is defined by $N = 12$ coordinates of the six nodes whereas the number of constraints (bars) is only $C = 11$.

Level 1. The given sizes suffice for evaluating the bar lengths and constructing the 11×12 constraint Jacobian matrix (Appendix, Table 1). Its transpose is the equilibrium matrix. The matrix rank $r = 10$, so the degree of statical indeterminacy of the system is $S = C - r = 1$. The initial (self-stress) force Λ^0_{10} in the central vertical bar is chosen as the independent one; thus, it determines the intensity of the single statically possible state of

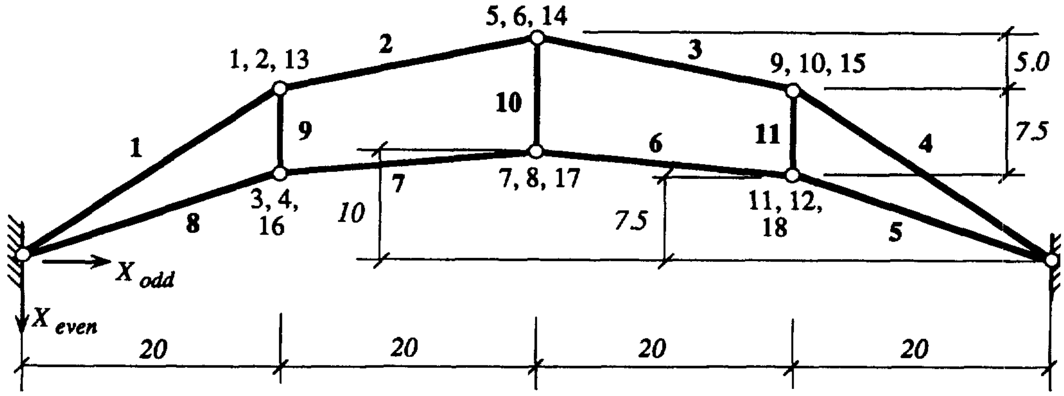

Fig. 1. Underconstrained (quasi-variant) pin-bar system.

self-stress for the system. This state is evaluated by: transposing the Jacobian matrix; removing any two rows from the obtained 12×11 equilibrium matrix (since $r = 10$); transferring the 10-th column to the right hand side; and solving the resulting systems of equations. In the end, the $C \times S$ (11×1) self-stress matrix b_{ik} (with $k = 10$ and $b_{10,10} = 1$) is constructed; its transpose, $b_{10,i}$, is

$$-2.5 \quad -2.062 \quad -2.062 \quad -2.5 \quad 4.272 \quad 4.031 \quad 4.031 \quad 4.272 \quad 1 \quad 1 \quad 1.$$

The unit force serves as the magnitude reference for all internal and external forces in the example.

Consider an equilibrium load $P_2 = P_6 = P_{10} = P = 1$ (Fig. 2) for which the upper chord of the system is a pressure polygon. As before, the statically determinate subsystem is obtained by disengaging bar 10. Solving the already available system of equilibrium equations (the one with column 10 removed), but this time with the given load in the right hand side, yields Λ_i^r:

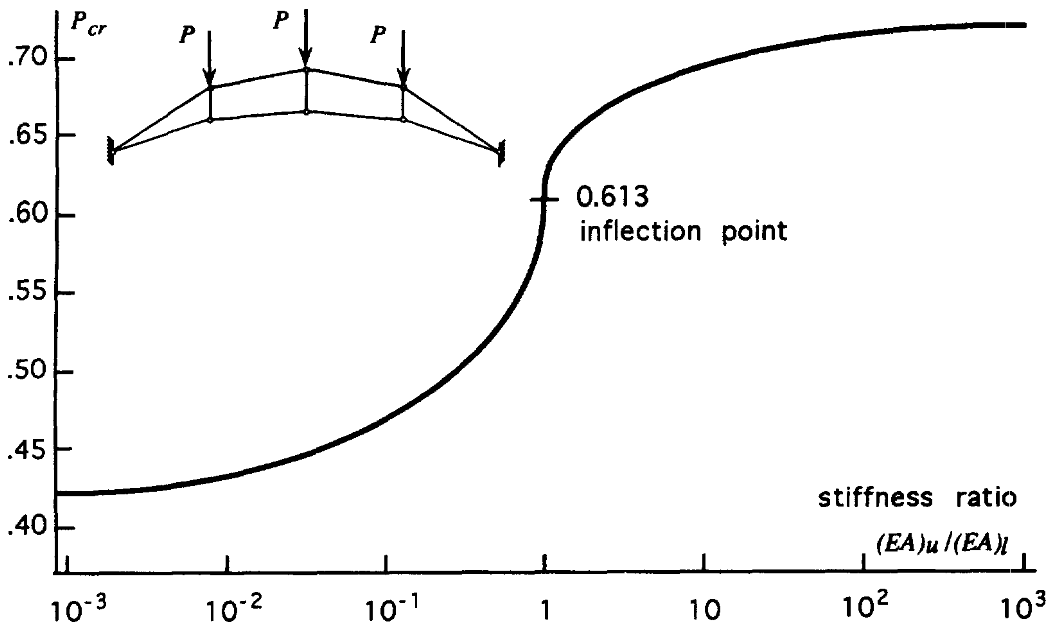

Fig. 2. Critical value of an equilibrium load vs bar stiffness ratio (semi-log scale).

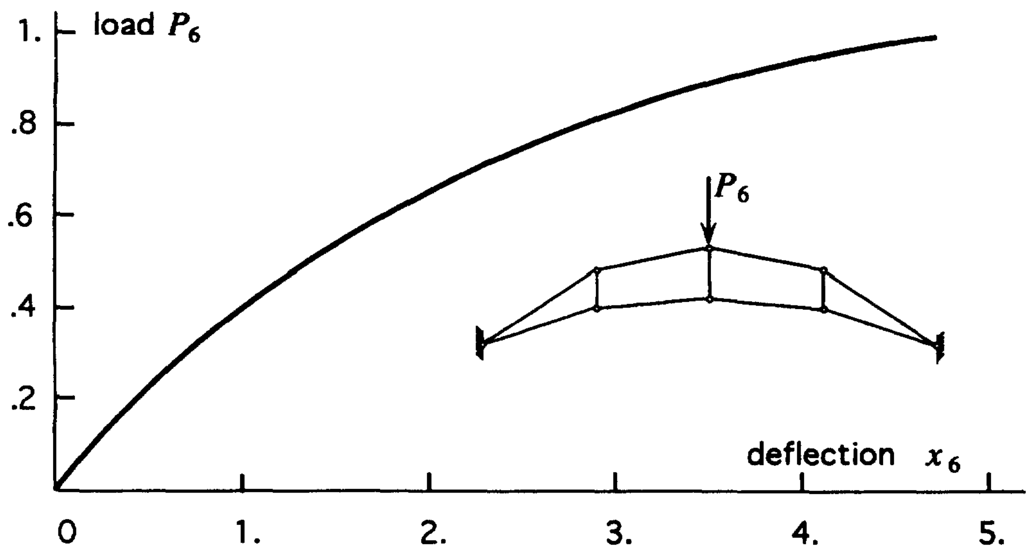

Fig. 3. Load–deflection relation for prestressed system under a general load.

$$-2.5 \quad -2.062 \quad -2.062 \quad -2.5 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0.$$

As expected, in this subsystem the load is balanced by the upper chord alone.

Setting k, $l = 10$ in (28) produces one equation in one unknown, the magnitude Λ_{10}^{P} of the redundant force. Assuming the vertical bars inextensible and the bars of the upper and lower chords having the same stiffness, $(EA)_u = (EA)_l = EA$, the respective elements of the diagonal flexibility matrix D_{ij} become 0 and L_i/EA. With this, eqn (28) yields the redundant force $\Lambda_{10}^{P} = -0.254$ and, according to (29), the bar forces produced by the applied load are

$$-1.866 \quad -1.539 \quad -1.539 \quad -1.866 \quad -1.084 \quad -1.023 \quad -1.023 \quad -1.084$$
$$-0.254 \quad -0.254 \quad -0.254.$$

This solution is exact because the external load is an equilibrium load. However, with all of the bars in compression, the stress state is likely to be geometrically unstable. If the system is prestressed, the final bar forces are obtained by combining this solution with the above state of self-stress; stability of the resulting equilibrium state remains uncertain.

For a general load, an exact solution does not exist; the best linear approximation is obtained either by separating and applying the equilibrium part (18) of the given load or directly, using eqn (11). For example, if the external load is a unit vertical force $P_6 = 1$ at the upper central node (Fig. 3), the approximate forces λ_i^r in the statically determinate subsystem are

$$-1.614 \quad -1.446 \quad -1.446 \quad -1.614 \quad 1.334 \quad 1.203 \quad 1.203 \quad 1.334 \quad 0.468 \quad 0 \quad 0.468.$$

Note that this time all of the subsystem bars are involved in balancing the equilibrium part of the applied load. The value of the redundant force found from (28) is $\Lambda_{10}^{P} = -0.397$. After evaluating the bar forces in the system due to the load, they are combined with the forces of prestress, and the resulting final bar forces are

$$-3.121 \quad -2.689 \quad -2.689 \quad -3.121 \quad 3.909 \quad 3.633 \quad 3.633$$
$$3.909 \quad 1.071 \quad 0.603 \quad 1.071.$$

The results obtained so far are purely formal. All of the stress states, including the assumed state of prestress, are just statically possible ones and, perhaps, unrealizable (if unstable). The solution accuracy, depending on the displacement magnitudes, is unknown, except for the case of equilibrium load and/or prestress, where the displacements are purely elastic and may be sufficiently small. Evaluating the system displacements and stability of equilibrium requires introducing the statical-kinematic stiffness matrix.

Level 2. The matrix K_{mn} (32) for the system is given in Table 2 (Appendix) using a shorthand notation; for example, the matrix element $K_{11} = \Lambda_1/L_1 + \Lambda_2/L_2 + \Lambda_9/L_9$ is presented as $1+2+9$. The matrix composition reflects the system connectivity. The constraint reactions Λ_i comprise the member forces due to both prestress and the equilibrium part of the applied load.

Going over to the reduced statical-kinematic stiffness matrix K_{st} (34) requires the virtual displacement matrix a_{ns}. The Jacobian matrix rank is $r = 10$, so that the number of independent displacements is $V = N - r = 2$. Taking advantage of the symmetry, the horizontal, x_5, and vertical, x_6, displacements of the upper central node are taken as independent. After transferring columns 5 and 6 of the Jacobian matrix to the right hand side and removing one of the rows, the resulting system of equations is solved for dependent displacements x_p. The $N \times V$ (12×2) virtual displacement matrix a_{ns} ($s = 5, 6$) of eqn (5) can now be formed; its transpose is

$$\begin{matrix} 1.5 & 2 & 0.75 & 2 & 1 & 0 & 0.5 & 0 & 1.5 & -2 & 0.75 & -2 \\ -0.375 & -0.5 & -0.188 & -0.5 & 0 & 1 & 0 & 1 & 0.375 & -0.5 & 0.188 & -0.5 \end{matrix}.$$

Note the segment 5, 6 in this matrix, produced by using the identity matrix I_{st}.

Upon constructing the matrix K_{mn} based on the prestress forces, the reduced statical-kinematic stiffness matrix K_{st} (34) is evaluated:

$$K^0_{5,6} = \begin{bmatrix} 1.525\Lambda^0_{10} & 0 \\ 0 & 0.481\Lambda^0_{10} \end{bmatrix}.$$

This matrix is important in two ways. First, its positive definiteness verifies stability of the state of self-stress, meaning that the actual prestress is feasible and the prestressed system will resist small perturbations. Second, the matrix quantifies the system resistance to perturbation loads depending on the prestress level and, implicitly, on the member stiffness ratios. For a given perturbation load (and only for such a load), the independent and then dependent displacements of the system are evaluated according to (35) and (5). (Recall that for a general load, the constraint reactions produced by its equilibrium part must be evaluated beforehand and incorporated into the K_{mn}.)

In the above solution for the equilibrium load, assume the load to be one-parametric. After scaling the found stress state by a load parameter P it is combined with the state of prestress and the obtained bar forces are introduced into K_{mn}. The resulting reduced stiffness matrix K_{st} is

$$K_{5,6} = \begin{bmatrix} 1.525\Lambda^0_{10} - 2.487P & 0 \\ 0 & 0.481\Lambda^0_{10} - 0.578P \end{bmatrix}.$$

From here, the critical load magnitude is found by determining the minimum value of P that leads to the loss of definiteness for the matrix:

$$P_{cr} = 0.613\Lambda^0_{10}.$$

The first critical load value happens to turn into zero the upper left element of the matrix, leaving the vertical displacement $x_6 = 0$, so that the corresponding bifurcation mode is antisymmetric.

Judging by the appearance of $K_{5,6}$, the system resistance to perturbations depends solely on the system geometry, prestress level and the magnitude of the equilibrium load.

In fact, as has been already noticed, in statically indeterminate systems, the elastic properties also affect the statical-kinematic stiffness matrix, although the effect is only implicit, through the member forces. Moreover, the member forces, hence, the matrix K_{st}, depend on the member stiffness ratios rather than the stiffness values. Thus, the system resistance to perturbations, as well as the critical load (the load magnitude at which this resistance vanishes) are also functions of the stiffness ratios. The graph in Fig. 2 shows the critical load P_{cr} plotted against the ratio $(EA)_u/(EA)_l$ of the upper and lower chords of the example system. At $(EA)_u = (EA)_l$ the curve has an inflection point.

The system stability as a plane structure does not imply the same for three dimensions. To investigate the spatial stability, six additional degrees of freedom (13 through 18, Fig. 1) are introduced for out-of-plane nodal displacements. (One of them must be constrained to preclude free-body rotation of the system about an axis passing through the support pins.) Both the 18×18 complete statical-kinematic stiffness matrix and the 18×8 virtual displacement matrix for the system in three dimensions can be constructed by adding new matrix blocks to the respective matrices for the plane system:

$$a_{\bar{n}\bar{s}} = \begin{bmatrix} a_{ns} & 0 \\ 0 & I_{n's'} \end{bmatrix}, \quad K_{\bar{m}\bar{n}} = \begin{bmatrix} K_{mn} & 0 \\ 0 & K_{m'n'} \end{bmatrix}, \quad n' = \bar{n} - 12, \quad s' = \bar{s} - 2.$$

Here the subscripts with bar indicate three-dimensional analysis; thus, $\bar{n}$ spans the range from 1 to $\bar{N} = N+6 = 18$ and the range size for $\bar{s}$ is $\bar{V} = V+6 = 8$. The added 6×6 lower right blocks are, respectively, the identity matrix and (taking advantage of symmetry of K_{mn} with respect to the X, Y and Z coordinates) a matrix assembled of rows and columns 1, 5, 9 and 3, 7, 11 of the original matrix K_{mn}.

Consider once again the system equilibrium under $P_2 = P_6 = P_{10} = 1$. It turns out that in three dimensions the system is stable only if all three upper nodes are restrained against out-of-plane displacements 13, 14 and 15. With the prestress magnitude of $\Lambda_{10}^{0} = 1$, the critical load value found above for the plane system is $P_{cr} = 0.613$. Taking $P = 0.600$, the reduced statical-kinematic stiffness matrix for the system in three dimensions has been evaluated:

5	0.0329	0	0	0	0
6	0	0.0742	0	0	0
16	0	0	0.4521	−0.1696	0
17	0	0	−0.1696	0.4239	−0.1696
18	0	0	0	−0.1696	0.4521.

The numbers in the first column designate the independent displacements. The matrix is positive definite, indicating stability of the partially restrained system in three dimensions.

As has been mentioned, one of the out-of-plane displacements always must be restrained. For some loads, this suffices for the system stability in 3 (and, hence, in 2) dimensions. Such is the case with constrained lower central node displacement $x_{17} = 0$ and a general load P_n given by

$$-3 \quad 0 \quad -1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 3 \quad 0 \quad 1 \quad 0.$$

After the member forces corresponding to the equilibrium part of the load have been evaluated and superimposed on the state of prestress, the familiar routine produced the following positive definite reduced statical-kinematic stiffness matrix (34) for the system in three dimensions:

5	2.2645	0	0	0	0	0	0
6	0	1.0847	0	0	0	0	0
13	0	0	0.0577	0	0	−0.1872	0
14	0	0	0	0.1082	0	0	0 .
15	0	0	0	0	0.0577	0	−0.1872
16	0	0	−0.1872	0	0	0.6124	0
18	0	0	0	0	−0.1872	0	0.6124

Level 3. Displacements of an underconstrained system with idealized (inextensible) members are evaluated by solving eqn (35) with a given general load in the right hand side. Consider the statical solution obtained earlier for the load $P_6 = 1$. The member forces due to the equilibrium part of the load and prestress are already known, making the evaluation of K_{st} straightforward. After calculating the right hand side in (35), the following system of equations is obtained:

$$\begin{bmatrix} 0.641 & 0 \\ 0 & 0.217 \end{bmatrix} \begin{bmatrix} x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Although the equilibrium is stable (positive definite K_{st}), the vertical displacement $x_6 = 4.610$ is too large compared to the system sizes, thus, for all practical purposes, invalidating the solution. Furthermore, it should be borne in mind that the obtained solution cannot be salvaged by a proportional scale-down of the load and displacement; this would be erroneous because of the above mentioned non-proportional relation between a general load and displacements in a system with preexisting internal forces. By segregating the member forces due to prestress from those due to the equilibrium part of the given load, the following system of equations is obtained from (35):

$$\begin{bmatrix} 1.525\Lambda_{10}^{0} - 0.890P_6 & 0 \\ 0 & 0.481\Lambda_{10}^{0} - 0.266P_6 \end{bmatrix} \begin{bmatrix} x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ P_6 \end{bmatrix}.$$

A graph of vertical displacement x_6 plotted against the load P_6 is shown in Fig. 3.

Level 4. The linear elastic analysis employing stiffness matrix (40) refines the inextensible displacements evaluated earlier. The latter are zero in an underconstrained system under an equilibrium load, so that in this case the newly obtained displacements are purely elastic. Assuming all of the bars in the example system having one and the same axial stiffness, EA, the found EA-multiples of the nodal displacements of the system are

22.62 76.65 13.88 76.65 0 64.84 0 64.84 −22.62 76.65 −13.88 76.65.

Elastic displacements under a general load with an appreciable perturbation component usually are much smaller than the inextensible displacements and may be disregarded. Accordingly, evaluation of the elastic displacements makes sense only if the found inextensible displacements are reasonably small.

CONCLUDING REMARKS

Comprehensive stiffness matrix (40) is the most general stiffness matrix for linear elastic analysis. Its coefficients involve terms of dissimilar nature, reflecting the various sources of first-order structural stiffness. In discrete bar systems such as frames, trusses and other pin-bar or cable systems, the first and predominant source of stiffness is the elastic axial stiffness of the bars. The next source is the elastic transverse stiffness of the bars in bending, shear and torsion. The third source is the statical-kinematic stiffness quantifying the resistance of a system with idealized members to perturbation loads. There exist some other, more exotic sources of first-order stiffness; one peculiar kind measures perturbation resistance of a statically indeterminate finite mechanism with an elastic interference caused by imperfect sizes or a change in temperature (Kuznetsov, 1991). The kinds of stiffness encountered in discrete structural systems have their counterparts in continuous systems such as plates, shells, membranes, fabrics and cable nets.

The various sources of stiffness can be ordered according to their typical magnitude thereby giving rise to the following stiffness hierarchy

	Typical order of magnitude		
Description of stiffness	Absolute	Relative	Numerical
Axial	EA/L	1	1
Transverse	EI/L^3	s^{-2}	10^{-4}
Statical-kinematic	N/L	ε	10^{-4}
Mechanism with interference	$\varepsilon_0 N_0/L$	ε_0^2	10^{-8}

Here the relative values are obtained by normalizing the respective absolute ones using the axial stiffness EA as a reference. Parameter s designates the bar slenderness ratio, ε is an elastic strain and ε_0 is the elastic strain due to interference.

The above comparison illustrates the relative significance of each kind of stiffness. For example, short of rendering the stiffness matrix singular or ill-conditioned, the transverse stiffness can be disregarded in the presence of the axial stiffness. This is why any bar system, with rigid or pin joints, is a truss as long as it can be analyzed as a truss. On the other hand, a singular or ill-conditioned matrix may be improved by incorporating the next stiffness on the list; such is the case with the example system of this paper. Moreover, if the joints between the compression members (the upper chord) are rigid, the system is no longer underconstrained; yet, since the transverse stiffness of the upper chord bars and the statical-kinematic stiffness of the system are of the same order, the latter should be introduced into the comprehensive stiffness matrix together with the former.

The presented analysis goes beyond of what most of the existing structural analysis software is set up to do. Typically, a pin-bar system of the kind considered here, even with an equilibrium load, is rejected, with some vague warning (usually complaining about the support conditions, although the apparent culprit is the singular stiffness matrix). Worse still, on some occasions an absurd numerical output is produced without warning. Both of these extreme responses can be avoided by employing the comprehensive stiffness matrix. Whereas the very necessity of its use should trigger an explanatory warning, the resulting analysis is always meaningful, even if not always conclusive.

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APPENDIX

Table A1. Constraint Jacobson matrix for the example system

0.8	−0.6	0	0	0	0	0	0	0	0	0	0
−0.970	0.243	0	0	0.970	−0.243	0	0	0	0	0	0
0	0	0	0	−0.970	−0.243	0	0	0.970	0.243	0	0
0	0	0	0	0	0	0	0	−0.8	−0.6	0	0
0	0	0	0	0	0	0	0	0	0	−0.936	−0.351
0	0	0	0	0	0	−0.992	−0.124	0	0	0.992	0.124
0	0	−0.992	0.124	0	0	0.992	−0.124	0	0	0	0
0	0	0.936	−0.351	0	0	0	0	0	0	0	0
0	−1	0	1	0	0	0	0	0	0	0	0
0	0	0	0	0	−1	0	1	0	0	0	0
0	0	0	0	0	0	0	0	0	−1	0	1

Table A2. Matrix K_{mn} for the example system

1+2+9	0	−9	0	−2	0	0	0	0	0	0	0
0	1+2+9	0	−9	0	−2	0	0	0	0	0	0
−9	0	7+8+9	0	0	0	−7	0	0	0	0	0
0	−9	0	7+8+9	0	0	0	−7	0	0	0	0
−2	0	0	0	2+3+10	0	−10	0	−3	0	0	0
0	−2	0	0	0	2+3+10	0	−10	0	−3	0	0
0	0	−7	0	−10	0	6+7+1	0	0	0	−6	0
0	0	0	−7	0	−10	0	6+7+10	0	0	0	−6
0	0	0	0	−3	0	0	0	3+4+11	0	−11	0
0	0	0	0	0	−3	0	0	0	3+4+11	0	−11
0	0	0	0	0	0	−6	0	−11	0	5+6+11	0
0	0	0	0	0	0	0	−6	0	−11	0	5+6+11